1. (a) Locate the critical point(s) of \( f(x) = xe^x \)

Solution:

\[
f'(x) = xe^x + e^x = (x + 1)e^x = 0 \quad \text{at} \quad x = -1.
\]

(b) Find the intervals on which \( f \) is increasing and decreasing.

Solution: \( f' \) is negative for \( x < -1 \), so \( f \) is decreasing on \((-\infty, -1)\). \( f' \) is positive for \( x > -1 \) so \( f \) is increasing on \((-1, \infty)\).

(c) Find the intervals on which \( f \) is concave up and concave down.

Solution: The second derivative is

\[
f''(x) = (1)e^x + (x + 1)e^x = (x + 2)e^x
\]

is negative for \( x < -2 \) and positive for \( x > -2 \). Therefore, \( f \) is concave down on \((-\infty, -2)\) and concave up on \((-2, \infty)\).

(d) Identify the local minimum and local maximum values of \( f \).

Solution: Using either the First or Second Derivative Tests, \( f \) has a local minimum value of \( f(-1) = -e^{-1} \).

(e) Identify the absolute minimum and maximum values of \( f \) on \([0, \frac{3}{4}]\).

Solution: Observe that our single critical points \( x = -1 \) is outside the domain of interest \([0, \frac{3}{4}]\). Comparing \( f(0) = 0 \) and \( f(3/4) = \frac{3}{4}e^{3/4} > 0 \). Therefore, \( f \) has an absolute minimum value of 0 and an absolute maximum value of \( (3/4)e^{3/4} \). With \( f(-1) = -\frac{2}{3} - 4 - 6 = -\frac{32}{3} \) and \( f(2) = \frac{8}{3} - 16 + 12 = -\frac{4}{3} \), we see that \( f \) has an absolute minimum of \(-\frac{32}{3}\) and absolute maximum of \(\frac{8}{3}\).

2. Use the second derivative test to find local minimum values of \( y = ex^2 - ex \). Give exact answers.

Solution: \( y' = 2ex - e \) and \( y'' = 2e > 0 \). There is a critical point when \( 2ex - e = 0 \), so \( x = 1/2 \). \( y''(1/2) = 2e > 0 \), so \( y \) is concave up at \( x = 1/2 \) which says \( y \) has a local minimum value at \( x = 1/2 \). The value equals \( e(1/2)^2 - e(1/2) = e(0.25 - 0.5) = -0.25e \).
3. Use L’Hôpital’s Rule to evaluate \( \lim_{x \to 1} \frac{x^2 - 2x + 1}{x^3 - 3x + 2} \).

**Solution:** Notice the limit is of indeterminate type \( \frac{0}{0} \).

\[
\lim_{x \to 1} \frac{x^2 - 2x + 1}{x^3 - 3x + 2} = \lim_{x \to 1} \frac{2x - 2}{3x^2 - 3}
\]

still has indeterminate type \( \frac{0}{0} \) so we use L’Hopital again to get

\[
\lim_{x \to 1} \frac{2}{6x} = \frac{2}{6} = \frac{1}{3}.
\]
4. Sketch a single function satisfying the following set of conditions:

\[
\begin{align*}
  f''(x) &< 0 \text{ on } (-8, 8), \\
  f'(x) &> 0 \text{ on } (-8, 0), \\
  f'(x) &< 0 \text{ on } (0, 8), \\
  \text{and } f(0) &= 0.
\end{align*}
\]

**Solution:** Answers may vary but require \( f \) to be concave down throughout \((-8, 8)\), increasing for \(-8 < x < 0\) and decreasing throughout \((0, 8)\), and passing through the point \((0, 0)\).
5. Does the Mean Value Theorem apply to the function \( f(x) = 3x^2 + x \) over the interval \([0, 2]\)? Why or why not? You must use complete sentences in your explanations. If the MVT does apply, find the value \( c \) it guarantees exists.

**Solution:** Yes, the MVT applies because \( f \) is continuous on \([0, 2]\) and differentiable on \((0, 2)\). The theorem guarantees some \( 1 < c < 3 \) with \( f'(c) = \frac{f(2) - f(0)}{2 - 0} \).

\[
6c + 1 = \frac{14 - 0}{2 - 0} = 7.
\]

The solution is \( c = 1 \) which, as expected is in the interval \([0, 2]\).

6. A turkey cage is to be built in the shape of a box with a square base. It is to have a volume of 120 cubic feet. The cedar wood for the base costs $50 per square foot, the stained-glass for the roof costs $100 per square foot, and the gold panels for the sides costs $30 per square foot. Find the dimensions of the most “economical” shed. If you have a calculator, please round your final numbers to the nearest tenth. If you do not have a calculator, then your answer will be algebraic expressions.

**Solution:** The objective function will be the cost, which we wish to minimize. To calculate the cost split up the individual costs of the base 50(base area), the roof 100(roof area), and the four sides 4 \cdot 30(area of sides). With the square base have side length \( x \) and the height being \( y \), the total cost is

\[
C(x, y) = 50x^2 + 100x^2 + 120xy = 150x^2 + 120xy.
\]

We eliminate one of the variables with the constraint equation \( V = 120 = x^2h \). This says that \( h = 120x^{-2} \). Substituting this into the cost function

\[
C(x) = 150x^2 + 120x(120x^{-2}) = 150x^2 + 14400x^{-1}.
\]

The derivative of cost (marginal cost) is \( 300x - 14400x^{-2} \). The critical point is

\[
300x^3 = 14400.
\]

\[
x^3 = \frac{14400}{300} = 48
\]

\[
x = \sqrt[3]{48} = 2\sqrt[3]{6} \text{ feet.}
\]

The other dimension is \( h = \frac{120}{(2\sqrt[3]{6})^2} = \frac{120}{4\sqrt[3]{36}} \) feet.
7. Of all numbers $x$ and $y$ whose difference equals $\pi$, find the two that minimize the quantity $x^2 + xy$. Give exact answers.

**Solution:** The constraint is $y - x = \pi$. Then $y = x + \pi$. Substituting that into the objective function yields

$$f(x) = x^2 + x(x + \pi) = 2x^2 + \pi x.$$  

Observe that the first derivative is $f'(x) = 4x + \pi$ switches from negative to positive at $x = -\pi/4$. This says $f$ has a local minimum there. The min. is absolute since we have only one local extrema. The numbers themselves are $x = -\pi/4$ and $y = x + \pi = 3\pi/4$.

8. Find an antiderivative of each of the following:

(a) $x^{1621}$

**Solution:** $\frac{1}{1622}x^{1622}$.

(b) $\cos(x)$

**Solution:** $\sin x$. 